ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, No. 4, 1978

TWO-GENERATOR FRATTINI SUBGROUPS OF FINITE p-GROUPS

BY

GAIL L. LANGE†

ABSTRACT

This paper deals with nonabelian p-groups T (p a prime and $p > 2$) which are either metacyclic or Redei. These groups are classified into those which are Frattini subgroups of a finite p-group G and those which are not. Finally, it is shown that a nonabelian two-generator group of order p^n ($n > 4$) which is the Frattini subgroup of a p-group must be metacyclic.

Introduction

The interrelationship between the structure of a finite p-group and that of the Frattini subgroup $\Phi(G)$ is examined. A group is metacyclic if there exists a cyclic normal subgroup whose factor group is cyclic. In Theorem 1 it is shown that if the Frattini subgroup of a p-group is a nonabelian metacyclic subgroup, then G contains a metacyclic subgroup B such that $\Phi(G) = \Phi(B)$. The author has classified all the nonabelian metacyclic groups as to whether they can be a Frattini subgroup or not.

The nonabelian Redei groups, which are those groups having all their maximal subgroups abelian, fall into two classes: metacyclic and nonmetacyclic. Theorem 2 states that no nonabelian nonmetacyclic Redei p-group can be a Frattini subgroup.

The author in [5] has checked all nonabelian groups of order $\leqq p^5$ for occurrence as a Frattini subgroup. The groups of order $\leqq p^4$ were done by Hobby in [2]. Of all the nonabelian groups of order p^5, nineteen could not be eliminated from being the Frattini subgroup by known results. We, in [5], eliminate thirteen of these. The other six can be Frattini subgroups and are for $p = 3$ (see [5]). Some of these results are generalized herein. It is shown that if T is any nonabelian two-generator group of order p^n ($n > 4$) such that $G' \leqq T \leqq$

†This work is contained in the author's dissertation.

Received November 2, 1976 and in revised form June 28, 1977

$\Phi(G)$ for a p-group G, then T must be metacyclic. Hobby had shown this for $T = \Phi(G)$ in [3].

THEOREM 1. *A nonabelian metacyclic p-group T is $\Phi(G)$ if and only if T is $\Phi(B)$ for a metacyclic subgroup B of G.*

PROOF. Set $\Phi_0 = \Omega_1(\Phi(G))$. Then $|\Phi_0| = p^2$. By [1] $G = AB$, for which A and B are defined as follows:

(1) A is generated by Φ_0 and all subgroups N of order p^3 and exponent p which contain Φ_0 and are normal in G,

(2) B is absolutely regular or of maximal class and $\Phi(G) \leqq B$,

(3) If B is absolutely regular, then $\Omega_1(B) = \Phi_0$,

(4) If B is not regular, then $p = 3$,

(5) A/Φ_0 is an elementary subgroup in the center of G/Φ_0,

(6) If $3 < p$, then A is regular and its exponent is p. Moreover, $A = \Omega_1(G)$ and G is regular.

First consider $p > 3$. Then B is regular and $\Omega_1(B) = \Phi_0$. From [4], it follows that B is metacyclic. Then $|B : \Phi(B)| = p^2$. Moreover, B normal in G implies $\Phi(B)$ is normal in G. The factor group $G/\Phi(B) = (A\Phi(B)/\Phi(B))(B/\Phi(B))$. Because $\Phi(B) \leqq \Phi(G) \leqq B$, three cases arise: (1) $\Phi(B) = \Phi(G)$; (2) $\Phi(G) = B$; (3) $\Phi(B) < \Phi(G) < B$ and each of index p in the next.

Case (2) has $G = A$ and $\Phi(G) = \Phi_0$. Hence, $\Phi(G)$ is abelian of order p^2. A contradiction arises. In Case (3) if $\Phi_0 \leqq \Phi(B)$, then $A\Phi(B)/\Phi(B)$ is abelian. But $A\Phi(B)/\Phi(B) \leqq Z(G/\Phi(B))$. Thus, $G/\Phi(B) = B/\Phi(B) \times A\Phi(B)/\Phi(B)$. So $G/\Phi(B)$ is the direct product of elementary abelian p-groups. Hence, $\Phi(G) \leqq \Phi(B)$. This contradicts the case under consideration.

Hence, $\Phi_0 \not\leq \Phi(B)$. Since $p > 2$, $\Phi(B)$ is a cyclic maximal subgroup in $\Phi(G)$ and $\Phi(G) = \Phi(B)\Phi_0 = \Phi(B)\langle a : a^p = 1\rangle$. But $\Phi_0 \leqq Z(\Phi(G))$ since $\Phi(G)$ is metacyclic and $Z(\Phi(G))$ cannot be cyclic [2]. Thus, $\Phi(G)$ is abelian. A contradiction again arises.

Thus, for $p > 3$ if $\Phi(G)$ is nonabelian, then $\Phi(G) = \Phi(B)$ and B is metacyclic.

Next, consider $p = 3$. By [1] $G = AB$ and B is regular or of maximal class. Suppose B is of maximal class. By [1] $B'' = 1$, B_1 is metacyclic and class B_1 is less than or equal to two. Also, $B' \leqq G' \leqq \Phi(G)$ and $|B : B'| = 3^2$. Since $B' = \Phi(B)$, then $\Phi(B)$ must be abelian. Hence, three possibilities exist: (1) $\Phi(G) = \Phi(B)$; (2) $B = \Phi(G)$; and (3) $|B : \Phi(G)| = 3$. Statement (1) implies $\Phi(G)$ is abelian. Statement (2) says B is metacyclic. Thus, B is regular and $|B| = 3^3$ by [4]. If B is a nonabelian 3-group of order 3^3, this contradicts (2) since $B = \Phi(G)$. Hence, B is abelian and $B = \Phi(G)$. A contradiction arises.

For Case (3) $|B:\Phi(G)| = 3$. From [1], $\Phi_0 \leqq B$. Hence, $\Phi_0 \leqq \Phi(B)$ or $\Phi(B) \leqq \Phi_0$ [4]. Consider $\Phi(B) \leqq \Phi_0$. Since $|\Phi_0| = 3^2$ and $|B:B'| = |B:\Phi(B)| = 3^2$, it follows that $|B| \leqq 3^4$. $|B:\Phi(G)| = 3$ implies $|\Phi(G)| \leqq 3^3$. Hence, $\Phi(G)$ must be abelian. A contradiction is obtained. Therefore, it is enough to consider $\Phi_0 \leqq \Phi(B)$. A/Φ_0 is elementary abelian and $A/\Phi_0 \leqq Z(G/\Phi_0)$. So $G/\Phi_0 = B/\Phi_0 \times C/\Phi_0$ such that $C \leqq A$ and C/Φ_0 is elementary abelian. Thus, $\Phi(G/\Phi_0) = \Phi(G)/\Phi_0 = \Phi(B)/\Phi_0$. Therefore, $\Phi(B) = \Phi(G)$. This is a contradiction to the case under consideration. Hence, B cannot be of maximal class.

If B is regular, then by the first part of this proof, T is $\Phi(B)$ for a metacyclic subgroup B of G.

A nonabelian metacyclic p-group T may be expressed as

$$T = \langle a, b : b^a = bb^{p^n}, b^{p^{n+k}} = 1, a^{p^m} = b^{p^{n+l}\lambda} \rangle.$$

All the possibilities for the relations among k, m, and n are now considered.

Corollary, *Let T be a nonabelian metacyclic p-group. Then*

(1) *If $m \leqq n$, $T \leqq \Phi(G)$, and $T \trianglelefteq G$ for a p-group G, then $k < m \leqq n$.*

(2) *If $1 < n$, $k < m$, and $k < n$, then T is the Frattini subgroup of a metacyclic p-group.*

(3) *Write $a^{p^m} = b^{\lambda_1 p^{n+t+1}}$ with $(\lambda_1, p) = 1$. If $k \geqq n$ then $T = \Phi(G)$ for a metacyclic p-group G if and only if $t \geqq k - n$, $k < m$ and $n > 1$.*

Theorem 2. *A Redei nonmetacyclic p-group cannot be a normal subgroup of a p-group G and contained in its Frattini subgroup.*

Proof. Let T be Redei metacyclic. Hence, $T = \langle a, b, c : a^{p^m} = b^{p^n} = c^p = 1, (a, b) = c \rangle$. Assume $T \leqq \Phi(G)$ and $T \trianglelefteq G$. Since $\mho_1(T) = \langle a^p, b^p \rangle$ is characteristic in T, then $\mho_1(T)$ is normal in G. Let $\bar{G} = G/\mho_1(T)$. Then $T/\mho_1(T)$ is a nonabelian group of order p^3 contained in $\Phi(G)/\mho_1(T) = \Phi(G)/\mho_1(T)$. This contradicts (2).

Corollary. *Let T be a nonabelian Redei metacyclic p-group of order p^n ($n > 4$), i.e., $T = \langle a, b : a^{p^{\alpha-1}} = b^{p^{\beta-1}} = 1$, $a^b = a^{p^{\alpha-2}}a$, $\alpha - 1 \geqq 2$, $\beta - 1 \geqq 1\rangle$. Then T can be a Frattini subgroup of a metacyclic group if and only if $\alpha - 1 \geqq 3$ and $\beta - 1 \geqq 2$.*

This concludes the classification of the nonabelian metacyclic and Redei groups which are the Frattini subgroup of a finite p-group. Next, we generalize the results on groups of order p^5 in [5] and Hobby's work in [3] where he dealt with $T = \Phi(G)$.

THEOREM 3. *Let T be a two-generator group of order p^n $(n \geqq 5)$ such that $G' \leqq T \leqq \Phi(G)$. Then T is metacyclic.*

PROOF. For p^5, it is shown in [5] that T must be a metacyclic Redei group of class two. For $|T| > p^5$, we consider two cases: class $T = 2$ and class $T > 2$. Let us consider the first case. Let T be a minimal counterexample for which the theorem is false. Let $|T| = p^s$. From [5], $s \geqq 6$. Let M be a subgroup of order p contained in $Z(G) \cap G'$. Then $G'/M \leqq T/M \leqq \Phi(G)M = \Phi(G/M)$, $|T/M| = p^{s-1}$, and T/M has two generators for otherwise T/M cyclic and $M \leqq Z(G)$ implies T abelian. There exists m in G so that $M = \langle m : m^p = 1 \rangle$.

First, assume T/M is nonabelian. Since class $T = 2$, class $T/M = 2$. Because $|T/M| < |T|$, T/M must be metacyclic. Then $T/M = \langle \bar{a}, \bar{b} : \bar{a}^{p^\alpha} = 1, \bar{b}^{p^\beta} = \bar{a}^{p^\gamma}, \bar{a}^{b} = \bar{a}^k \rangle$ with $k^p \equiv 1 \pmod{p^\alpha}$, $p^\gamma(k-1) \equiv 0 \pmod{p^\alpha}$, and $\alpha + \beta = s - 1$ (4, Satz. 11.2]. Hence, $T = \langle a, b, m : m^p = 1, a^{p^\alpha} = m^i, b^{p^\beta} = a^{p^\gamma} m^j, a^b = a^k m^l$ for integers i, j, and l. If $m \not\in \mho_1(T)$, then $T/\mho_1(T)$ is a nonabelian group of order p^3 contained in $\Phi(G/\mho_1(T))$. This contradicts (2). Hence, $m \in \mho_1(T)$ and $\Phi(T) = \mho_1(T)$ has index p^2 in T. Thus, T is metacyclic.

Secondly, assume T/M is abelian. Since T has two generators, there exist a and b in G so that $T/M = \langle \bar{a} \rangle \times \langle \bar{b} \rangle$. Hence, $T = \langle a, b, m : m^p = 1, a^{p^u} = m^i, b^{p^v} = m^j, a^b = am^k \rangle$ for integers u, v, j, and k such that $k \not\equiv 0 \pmod{p}$ and $u + v = s - 1$. If $m \not\in \mho_1(T)$, then a contradiction to (2) is again obtained. Thus, $\Phi(T) = \mho_1(T)$ and T is metacyclic.

For Case (2), see [5].

REFERENCES

1. Ja. Berkovic, *A generalization of the theorems of Hall and Blackburn and their applications to nonregular p-groups*, Math USSR-Izv. **5** (1971), 829–832.

2. C. Hobby, *The Frattini subgroup of a p-group*, Pacific J. Math. **10** (1960), 209–212.

3. C. Hobby, *Generalizations of a theorem of N. Blackburn on p-groups*, Illinois J. Math. **5** (1961).

4. B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

5. G. Lange, Doctoral dissertation, University of New Hampshire, 1975.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MAINE
FARMINGTON, MAINE 04938 USA